Abstracts of the MAnET Workshop

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Samaneh Abbasi Sureshjani, Eindhoven University of Technology

Automated Analysis of Retinal Images for Early Diabetes Detection with Sub-Riemannian Methods

Abstract: Recent studies show that cardiovascular and systematic diseases such as diabetes are reaching epidemic proportions, particularly in developing countries. China is now among the countries with the highest diabetes prevalence in Asia (10%!). Diabetic retinopathy (DR) is the result of progressive damage to the network of tiny blood vessels that supply blood to the retina, and it is the leading cause of vision loss in working adult populations. Therefore, it is essential to diagnose and control DR at early stages.

Quantitative geometric analysis of the retinal vasculature and background is very useful for diagnosis, analysis and treatment of diabetes at different stages. The RetinaCheck project is an international collaboration between Eindhoven University of Technology (TU/e), Maastricht University and a number of clinical partners in Shenyang, China. The main goal is to build a Computer Aided Diagnosis (CAD). The main focus of the RetinaCheck team at TU/e is analyzing the retinal vasculature. Segmenting, enhancing and delineating the tiny vessels at low-contrast regions and noisy images is very challenging. However, existing methods encounter great difficulties when dealing with crossovers commonly seen in vessel networks.

Motivated by recent findings about the orientation-selective property of the receptive fields in the primary visual cortex (V1), orientation scores are constructed by lifting all elongated structures (in 2D images) along an extra orientation dimension. Bekkers et al. developed a fully automatic and robust multi-orientation vessel tracking algorithm. Zhang et al. showed that by applying left-invariant diffusion on the invertible orientation score of a 2D image, the elongated structures can be excellently enhanced without destroying the separated crossing parts in the score domain. Hannink et al. introduced a new vesselness filter via scale-orientation scores.

Very few studies have proposed delineation techniques for images taken with scanning laser ophthalmoscope (SLO) cameras. A comparison with color images is needed for appropriate (crossing preserving) enhancement, denoising, delineation and tracking techniques for these images.

Andrea Calogero, University of Milano-Bicocca

Sharp Alexandrov Maximum Principle via Horizontal Normal Mapping in the Heisenberg Group

Abstract: In this talk, first we solve a problem raised by Gutiérrez and Montanari about comparison principles for H-convex functions on subdomains Ω in the Heisenberg group $\mathbb{H}$.

Our approach is based on the notion of the sub-Riemannian horizontal normal mapping $\partial_H u$ of the function $u : \Omega \to \mathbb{R}$. More precisely, one defines the horizontal subdifferential $\partial_H u(\xi_0)$ of u at $\xi_0 \in \Omega$ given by

$$\partial_H u(\xi_0) = \left\{ p \in \mathbb{R}^2 : u(\xi) \geq u(\xi_0) + p \cdot (\mathrm{Pr}_1(\xi) - \mathrm{Pr}_1(\xi_0)),\ \forall \xi \in \Omega \cap H_{\xi_0} \right\},$$

where Pr_1 is the projection $\mathrm{Pr}_1(\xi) = \mathrm{Pr}_1(x, y, t) = (x, y)$ and $H_{\xi_0} \subset \mathbb{H}$ is the horizontal plane in $\xi_0 \in \mathbb{H}$. The range of the horizontal normal mapping of the function u is defined by

$$\partial_H u(\Omega) = \bigcup_{\xi \in \Omega} \partial_H u(\xi);$$

this notion is introduced by Danielli, Garofalo and Nhieu and studied by Calogero and Pini. Using the degree theory for set-valued maps we prove our first result

[Comparison principle for the horizontal normal mapping] Let $\Omega \subset \mathbb{H}$ be an open, bounded and H-convex set, and $u, v : \Omega \to \mathbb{R}$ be H-convex functions. Let $\Omega_0 \subset \mathbb{H}$ be open such that $\overline{\Omega_0} \subset \Omega$ and assume that $u < v$ in Ω_0 and $u = v$ on $\partial\Omega_0$. Then

$$\partial_H v(\Omega_0) \subset \partial_H u(\Omega_0).$$

This statement of the comparison principle combined with a Harnack inequality is applied to prove the second and main result of the talk:

[Aleksandrov-type maximum principle] Let $\Omega \subset \mathbb{H}$ be an open, bounded and convex set. If $u : \overline{\Omega} \to \mathbb{R}$ is a continuous H-convex function which verifies $u = 0$ on $\partial\Omega$, then

$$|u(\xi_0)|^2 \leq C \mathrm{dist}_H(\xi_0, \partial\Omega)\ \mathrm{diam}_H(\Omega) L^2(\partial_H u(\Omega)), \qquad \forall \xi_0 \in \Omega,$$

where $C > 0$ is a constant (dist_H and diam_H denote horizontal distance and diameter respectively).

This result describes the correct boundary behavior of continuous H-convex functions vanishing at the boundary of bounded and convex subdomains of $\mathbb{H}$. This Aleksandrov-type maximum principle is really the sub-Riemannian version of the well known result in the Euclidean setting and answers a question by Garofalo and Tournier. The sharpness of our results are illustrated by examples.

These recent results are obtained in a long and fantastic collaboration with Z. M. Balogh and A. Kristály.

Vasilis Chousionis, University of Helsinki
Removability on Carnot Groups

Abstract: A fundamental problem in partial differential equations is to understand the size and geometric structure of removable sets for solutions to a given PDE. Geometric Measure Theory provides the necessary machinery in order to quantify and geometrically describe such removable sets. In the Euclidean case classical results have been obtained by various authors including Carleson, David, Mattila, Harvey and Polking. This talk will focus on recent work which extends several of these results in the setting of Carnot groups. I will also discuss how new singular integrals emerge naturally in our proofs, motivating several open questions in geometric harmonic analysis.

Katrin Fässler, University of Bern
Quasiconformal Nonequivalence of the Heisenberg Group and the Roto-translation Group

Abstract: This talk is concerned with two Lie groups modeled on $\mathbb{R}^3$ and endowed with a sub-Riemannian structure: the Heisenberg group and the (universal cover of the) roto-translation group. The resulting metric spaces have the same tangent cones, but, as we will see by means of geometric methods, they are not quasiconformally equivalent. More generally, the proof idea yields sufficient conditions for two metric measure spaces of bounded geometry to be quasiconformally nonequivalent. (This is joint work with P. Koskela and E. Le Donne.)

Matteo Galli, University of Bologna
Sub-elliptic PDE Related with Curvature in Sub-Riemannian Manifolds and Applications to Vision

Abstract: We discuss the study of certain sub-elliptic PDE related with the mean curvature of surfaces. In particular we are interested in the study of:

1) curvature flow of curves and surfaces;

2) relation between heat flow in the whole space and induced evolution of submanifolds;

3) stability operator for minimal surfaces;

4) application to Vision.

Cristian Gutiérrez, Temple University
Lectures on Monge-Ampère Equations and Applications

Abstract: Monge-Ampère type equationsinvolve the Jacobian determinant of a map, and arise in the mathematical description of numerous geometric, optical, acoustic, and electromagnetic applications; in particular, in lens and reflector antenna design and in optimal mass transportation. The lectures present these equations and some recent applications to solve problems in geometric optics. I will try to emphasize ideas and techniques applicable to other contexts. Tentatively the topics to cover are the following:

1. Generalized solutions to the MA equation and basic properties.

2. Basic facts about optimal mass transport.

3. Geometrical optics; description of the far field and near field refraction problems and overview of their solution using the methods from (1) and (2) above.

References:

C. E. Gutiérrez. The Monge–Ampère equation. Birkhäuser, Boston, Mass., 2001.

C. E. G. and Qingbo Huang. The refractor problem in reshaping light beams. Arch. Rational Mech. Anal., 193(2), pp. 423–443, 2009.

C. E. G. and H. Mawi. The refractor problem with loss of energy. Nonlinear Analysis: Theory, Methods & Applications, vol. 82, pp. 12-46, 2013.

C. E. G. and Q. Huang. The near field refractor. Annales de L'Institut Henri Poincaré (C) Analyse Non Linéaire, vol. 31 (4), pp. 655-648, 2014.

Piotr Hajlasz, University of Pittsburgh
The Heisenberg Groups

Abstract: The purpose of the course is an introduction to the geometric theory of the Heisenberg groups. In particular I will discuss the metric structure of the Heisenberg group, the Hausdorff dimension, the structure of geodesics, unrectifiability and Lipschitz homotopy groups. The course will be self-contained and accessible to graduate students who are new to the subject.

Alexandru Kristaly, Babes-Bolyai University
Functional Inequalities in Metric Measure Spaces with Applications

Abstract: In this talk we shall discuss some functional inequalities (Caffarelli-Kohn-Nirenberg inequality, Gagliardo-Nirenberg inequality and their limit cases) in the framework of metric measure spaces. As application, certain rigidity results will be provided on Riemannian and Finsler manifolds depending on their curvature.

Enrico Le Donne, University of Jyvaskyla
Isometries of Groups and Homogeneous Spaces

Abstract: We consider homogeneous spaces equipped with geodesic distances. We are interested in studying the isometries of such metric spaces. This study will provide a metric characterization of Carnot groups. The work is in collaboration with L. Capogna and A. Ottazzi.

Valentino Magnani, University of Pisa
Perimeter Measure in Carnot Groups and the Metric Unit Ball

Abstract: Motivated by recent results in Geometric Measure Theory, we present a new method to compute the perimeter measure in terms of the spherical Hausdorff measure. We discuss how the geometry of the metric unit ball plays a prominent role.

Alexey Mashtakov, Eindhoven University of Technology
Sub-Riemannian Problems on 3D Lie Groups with Applications to Retinal Image Processing

Abstract: Two problems will be discussed:

(A) a sub-Riemannian (SR) problem in SE(2) with given external cost,

(B) a sub-Riemannian (SR) problem in SO(3) with cuspless spherical projection constraint.

Both problems have applications in image analysis, namely their solutions provide a method for completion of corrupted contours in flat and spherical images. The restored arc given by solution of (A) represents a compromise between the sub-Riemannian geodesics in SE(2) and horizontal curves minimizing the integral of an external cost induced by the image. Problem (B) is the spherical extension of problem (A). Solution to these problems are applied to retinal vessel tracking, relevant for early diagnosis of diabetic retinopathy and glaucoma.

(A) We propose an algorithm for solving the sub-Riemannian problem with external cost C in SE(2) consisting of two steps. The first step is producing the distance map in SE(2) while the second step is finding a solution to the BVP via steepest descent backtracking. The distance map is computed as the stationary viscosity solution of the corresponding Hamilton-Jacobi-Bellman (HJB) equation. An upwind finite differences discretization scheme is used for solving the HJB equation while the steepest descent is computed by numerical integration of the horizontal part of the Pontryagin Maximum Principle. The comparison with the exact solution in case $C = 1$, that was given by Yu.L. Sachkov(2011), demonstrates the accuracy of this implementation. Finally, we show applications of vessel tracking in retinal images. (Joint research with E.J. Bekkers, R. Duits and G.R. Sanguinetti).

(B) The problem of finding SR-geodesics in SO(3) with cuspless spherical projections is stated as follows: find a smooth curve with fixed boundary points and directions on a unit sphere, minimizing the compromise among length and geodesic curvature. This problem is a natural spherical extension of the cortical model by J. Petitot(2003), G. Citti and A. Sarti (2006). Our motivation is the fact that the retina is not flat, which is important both for cortical modeling and for processing retinal images. We derive stationary curves and evaluate the first cusp time, i.e. the instance of time where the spherical projection of SR-geodesic has a cusp. Parametrization of the set of endpoints reachable by geodesics with cuspless spherical projections is provided, as well as investigation of the absence of conjugate points in asymptotic case. In contrast to SR-problem in SE(2) there exist non-optimal SR-geodesics with cuspless projections. (Joint research with R. Duits, Yu.L. Sachkov and I. Beschastnyi).

Joan Mateu, Autonomous University of Barcelona
Rotating Vortex Patches for Euler Equation

Abstract: The purpose of this talk is to present the Euler equation, the vorticity equation and the vortex patches problem for planar, incompressible and inviscid fluids. We will explain that some vortex patches rotates and we will give a theorem of existence of rotating vortex patches for simply connected domains and for doubly connected domains.

Roberto Monti, University of Padova
Minimal Surfaces and Harmonic Functions in the Heisenberg Group

Abstract: One of the central facts in the regularity theory of minimal currents in R^n is the existence of a harmonic function in the blow-up of the boundary, rescaled by excess in the graph direction. In this talk, we investigate the possibility of a similar phenomenon in the Heisenberg group H^n with $n \geq 2$. We show that, for sets that are H-perimeter minimizing in a stronger sense, the blow up of the boundary contains H-harmonic functions for the Kohn Laplacian of a lower dimensional Heisenberg group.

Pierre Pansu, Paris-Sud University
The Quasi-symmetric Hölder Equivalence Problem

Abstract: What is the best Hölder continuity exponent for a homeomorphism of Heisenberg group to Euclidean space? Riemannian geometry suggests a variant of this natural question, where (easier) the inverse homeomorphism is assumed to be Lipschitz, and where (harder) Heisenberg group is replaced with an arbitrary metric space quasi-symmetric to Heisenberg group. We give a partial result on this question.

Ioannis Platis, University of Crete
Hyperbolic Heisenberg Group

Abstract: The usual Heisenberg group H is $C \times R$ with a group multiplication which turns it into a 2 step nilpotent group. In the study of quasiconformal mappings in this setting, mappings which preserve the vertical axis V=0X R frequently arise. These are comparable to the usual qc mappings of the plane that preserve the real line. Therefore it is natural to ask if there is a structure in $H* = C \times R$ minus the axis V which has an interest of its own. It turns out that indeed this is the case and in this talk we will describe some of the properties of H*. Among others we will show that:

1. There is a group multiplication such that H* is a 3 dimensional Lie group (not a Lie subgroup of H and not nilpotent).

2. There is a Riemannian metric on H*; with this metric H* is isomorphic to the unit tangent bundle of the usual hyperbolic plane.

3. Contactomorphisms of H restricted to H* have an additional special property.

Alessandro Sarti, The School for Advanced Studies in the Social Sciences
The Sub-Riemannian Structure of the Visual Cortex: Ten Years After

Abstract: First of all I will recall the original models of the functional architectures of the primary visual cortex as the Euclidean group $SE(2)$ equipped with a sub-Riemannian metric. Extensions to different Lie groups have been proposed later to take into account a multiplicity of visual features. Finally different PDEs in this geometric structure have been proposed to model fundamental problems of visual perception like the constitution of perceptual units, the problem of inpainting and the modal completion of the Kanizsa triangle. A review of these models will be performed. Joint work with Giovanna Citti

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Francesco Serra Cassano, University of Trento
Non-parametric Minimal Surfaces in Heisenberg Groups

Abstract: In the setting of the sub-Riemannian Heisenberg group $\mathbb{H}^n \equiv \mathbb{R}^{2n+1}$ we will introduce the classes of t- and intrinsic graphs of bounded variation. We will deal with the existence of non-parametric minimal surface, i.e. of graphs with the least possible area among those with the same boundary. For some minimal t-graphs we will also provide a uniqueness result. Eventually we will discuss some open problems.

Stefan Wenger, University of Fribourg
Lipschitz Homotopy Groups of the Heisenberg Groups

Abstract: The n-th Heisenberg group H_n, when endowed with its sub-Riemannian metric, is homeomorphic to Euclidean space R^{2n+1}, however not even locally biLipschitz homeomorphic to it. We study the question for which integers k Lipschitz maps from the Euclidean k-sphere to H_n extend to Lipschitz maps on the $(k+1)$-ball. When k¡n such extensions always exist. When $k = n$, however, no "non-trivial" Lipschitz map from S^n to H_n admits an extension. The situation is more complicated and not yet well-understood for $k > n$. In this talk I will describe what is known about this problem. In particular, I will answer in the negative the following question, recently raised by DeJarnette-Hajlasz-Lukyanenko-Tyson: is it true that the composition of a homotopically non-trivial Lipschitz map from S^k to S^n with a biLipschitz embedding of S^n into H^n does not admit a Lipschitz extension to the $(k+1)$-ball? Based on joint work with Robert Young.

Kevin Wildrick, Montana State University
Steiner's Formula for the Carnot-Carathéodory Distance in the Heisenberg Group

Abstract: Steiner's formula states that the volume of an epsilon neighborhood of sufficiently smooth set in n-dimensional Euclidean space is a polynomial of degree n, whose coefficients carry information about the curvature of the boundary of the set. We will provide an analogous result for the Carnot-Carathéodory distance in the first Heisenberg group. Although the resulting function is not, in general, a polynomial, it is analytic and the coefficients in its series expansion are integrals of second order differential operators. In particular, this approach produces a candidate for the notion of "horizontal Gauss curvature" in the Heisenberg group.

Roger Züst, Paris Diderot University
Some Results on Maps That Factor Through a Tree

Abstract: We give a necessary and sufficient condition for a map defined on a quasiconvex and simply-connected space to factor through a tree. In particular if the target is the plane and the map is Hölder continuous with exponent bigger than 1/2, such maps can be characterized by the vanishing of some integrals over the winding number function. Moreover, this shows that if the target is the Heisenberg group equipped with the Carnot-Carathéodory metric and the Hölder exponent of the map is bigger than 2/3, the map factors through a tree.